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DEFLECTIONS IN GRIDWORKS AND SLABS

By Walter W. Ewell, Jun. ASCE, Shigeo Okubo, and Joel I. Abrams, Jun. ASCE

STRUCTURAL DIVISION

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PAPERS

DEFLECTIONS IN GRIDWORKS AND SLABS

By Walter W. Ewell,[1] Jun. ASCE, Shigeo Okubo,[2] and Joel I. Abrams,[3] Jun. ASCE

Synopsis

Because the exact determination of elastic deformations in plates and slabs by the mathematical theory of elasticity has proved to be impractical in all but the simple cases, engineers have been forced to devise approximate means of solution. A horizontal gridwork of beams intersecting at right angles, with deformation characteristics analogous to those of a slab, will develop an elastic surface under normal load similar to the surface of the slab under the same load. A technique is presented to determine the elastically deformed surface of a plate or slab that has been transformed into an equivalent grid system. The method employs an auxiliary force system for controlling vertical displacements of the joints and a moment and torque distribution process for transmission of the displacement effects.

Introduction

Approximate methods of determining the deflected surfaces of slabs are not new. A system of beams, intersecting at right angles to one another, forms a gridwork that yields a deflected surface similar to that of a slab when analyzed under normal loads. This statement appears obvious when one considers the similar elastic surfaces developed by a slab and by a simple window screen supported and loaded in like fashion. The analysis of such a gridwork is not simple, however.

One method of determining the deflection in a gridwork of beams is presented by Miklós Hetényi,[4] who assumes that the individual beams comprising the gridwork deflect without rotation at their intersections with other beams.

Note.—Written comments are invited for publication; the last discussion should be submitted by March 1, 1952.

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[4] "Beams on an Elastic Foundation," by Miklós Hetényi, The Univ. of Michigan Press, Ann Arbor, Mich., 1946, pp. 185–192.

Simultaneous differential equations are then solved in order to obtain joint displacements.

When it is assumed that no bending or twisting moment is transmitted at grid beam intersections, another method of grid analysis is possible. With such an assumption, Stephen S. Timoshenko employs trigonometric series to express the elastic curves developed by the individual grid beams.[5] A gridwork, unable to transmit bending and torsional moments, is not, however, analogous to a slab or plate in its action.

If a realistic rigid joint, capable of resisting bending and torsion, is assumed at each beam intersection, the analysis of the resulting grid is a tedious process. Linear equations can be formed involving three unknown elements at each joint—the loads on the beam, the moment change in the longitudinal beam, and the moment change in the transverse beam.[6] Both moment changes are developed by torsional effects in the mutually perpendicular beams. An inordinate amount of time is involved in the solution of the simultaneous equations particularly since there are three times as many equations as there are joints in the grid.

A technique is here presented for eliminating two of the three unknown quantities at each joint by a moment and torque distribution process. In some of its aspects, this procedure is similar to the method of finite differences developed by Henri Marcus[7] and extended to skew slabs by Vernon P. Jensen.[8]

Finite difference equations have been applied to rectangular and skew slabs divided into a network of points determined by arbitrarily ruling the surface with a series of intersecting lines. Moments and deflections are obtained at each point of the network by solving simultaneously two groups of linear equations: one group containing moment values as unknowns, the second group expressing deflection values as unknowns. The number of equations to be solved in one group is, in the general case, equal to the number of points in the arbitrary network. The solution of the two groups of simultaneous equations can be avoided if the original fourth order differential equation of the plate is not divided into two parts. When finite difference equivalents are determined directly from the fourth order equation, a solution is required for only one series of linear equations. Nevertheless, this procedure is not recommended by Mr. Marcus.

In comparison, the method used in this paper is applicable to a physically analogous grid rather than to the ruled surface of the slab itself. After the bending moments and torsional moments have been distributed over the grid, only one series of linear equations need be solved in order to define the deflection pattern. This is possible because the equations are written in terms of unknown deflections produced by auxiliary forces at the grid points. Bending moments and torsional moments can then be found without recourse to the

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[5] "Über die Biegung von Trägerrosten," by Stephen S. Timoshenko, *Zeitschrift für angewandte Mathematik und Mechanik*, Band 13, 1933, p. 153.

[6] "Trägerrost und Platte," by Béla Enyedi, *Beton und Eisen*, Vol. 34, Heft 3, 1935, p. 42.

[7] "Die Theorie elasticher Gewebe und ihre Anwendung auf die Berechnung biegsamer Platten," by Henri Marcus, Julius Springer, Berlin, 1924.

[8] "Analysis of Skew Slabs," by Vernon P. Jensen, *Bulletin Series No. 332*, Univ. of Illinois, Urbana, Ill., 1941.

second series of simultaneous equations that are required in the simplified finite difference solutions.

Procedure

The procedure for determining the elastic surface of a plate or slab subjected to normal loading consists of the following steps:

1. The arbitrary division of the slab into orthogonal strips that are considered the beams of the analogous gridwork.
2. The determination of factors, based on the individual beam's resistances to bending and to twist, for distributing the unbalanced moments at each grid beam joint.
3. The vertical displacement of each singular joint in the grid and consequent introduction of fixed-end moments on the beams. Joints where deflection produces a distinctive moment and torque pattern are known as singular joints.
4. The conversion of the fixed-end moments into bending and twisting moments by a moment and torque distribution process.
5. The computation of reactions at all joints in terms of the unknown displacements.
6. The computation of the moments and torques at each joint which were expressed by the distribution process in terms of the arbitrary displacements.

Transformation of the Slab into a Comparable Grid.—Any plate or slab, when divided into strips in two mutually perpendicular directions, can be transformed into a grid system provided that each strip be considered as a grid beam. In fact, an analogy between slab and gridwork is indicated by the close correspondence of slab deflection values found by the grid method and checked by the theory of elasticity. Obviously, a finer division of the slab will increase the number of grid beams and will lead to a more accurate definition of the elastic surface.

If the slab shown in Fig. 1 is divided into two series of parallel orthogonal strips, the resulting system simulates a group of rigidly connected continuous beams mutually supported along their centerlines. For example, the east-west and north-south grid strips are considered continuous over supports 1–1, 2–2, 3–3, and over supports 1′–1′, 2′–2′, 3′–3′, located at the centerlines of the two perpendicular strips. Furthermore, a beam segment such as 1–2–2–1 is assumed free to rotate and to twist over its supports. Along the edges 1–2 the resistances to bending and twist that are presumably neglected in the east-west beams are actually considered as vertical shear forces in the north-south beams. Because the cross section of any grid beam is identical to that of a comparable slab strip, the gridwork is actually a pseudogrid. The over-all general appearance of the system is no different from that of the slab because no openings exist between grid beams.

Since the equivalent gridwork is to be loaded and consequently subjected to moments and torques, relative stiffnesses and distribution factors of the individual grid beams must be calculated.

Relative Stiffnesses and Distribution Factors.—If an applied moment (M), in the plane of the beam A 1 C, is released on the rigidly connected system shown in Fig. 2, the tangent to the elastic curve will rotate through an angle θ. If torsional fixity is realized at points D and B, the beam D 1 B will twist through the same angle θ at point 1. Similarly, a moment applied in the plane D 1 B will produce rotation of the tangent to the elastic curve in that plane and consequent twist in the beam A 1 C.

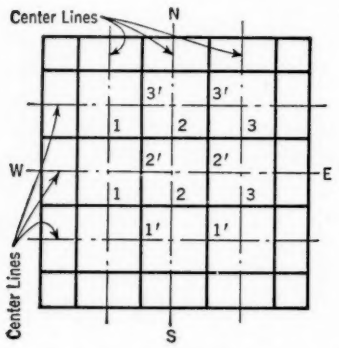


FIG. 1.—DIVISION OF SLAB

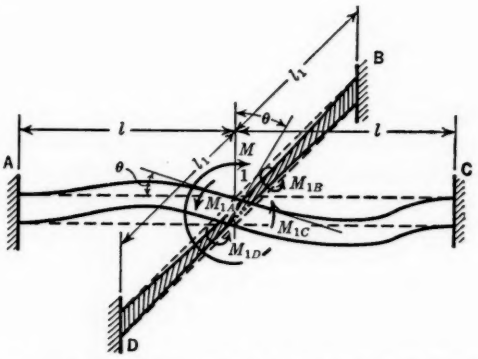


FIG. 2.—BENDING AND TWISTING OF A ROTATED RIGID JOINT

In order to resist the applied moment M, bending moments M_{1A} and M_{1C} will be developed in beam A 1 C, and torsional moments M_{1D} and M_1B will be introduced in beam D 1 B.

When the rotation through the angle θ has taken place,

$$M = M_{1A} + M_{1C} + M_{1D} + M_{1B} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

For a unit rotation θ in beam A 1 C, using slope deflection equations,

$$M_{1A} = \frac{4EI}{l} \qquad \text{and} \qquad M_{1C} = \frac{4EI}{l} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

For a unit twist θ in the beam D 1 B, with circular cross section:

$$M_{1D} = \frac{GI_p}{l_1} = M_{1B} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3a)$$

with rectangular cross section, or,

$$M_{1D} = \frac{\beta c b^3 G}{l_1} = M_{1B} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3b)$$

in which c = width of grid beam; b = depth of beam; G = modulus of elasticity in shear; I_p = polar moment of inertia of beam or grid strip; I = moment of inertia with respect to an axis perpendicular to the plane of bending; β = coefficient depending on cross-sectional properties[9]; and E = modulus of

[9] "Elements of Strength of Materials," by S. Timoshenko and Gleason H. MacCullough, D. Van Nostrand Co., Inc., New York, N. Y., 1949, pp. 265–266.

elasticity. With the expressions for moment in the longitudinal and transverse beams known, distribution factors can be written for any joint. For example:

$$M_{1A} = \frac{4EI/l}{4EI/l + 4EI/l + GI_p/l_1 + GI_p/l_1} \quad \ldots\ldots\ldots\ldots (4)$$

Distribution factors for the ordinary grid will be the same at every point when lengths between intersections are made constant in each direction and identical cross sections are selected.

Auxiliary Force Systems for Controlling Deflections.—The moment distribution process to be used cannot be applied directly to any loaded structure whose joints undergo translation. In order to control or prevent these joint translations, it is necessary to calculate a series of auxiliary force systems that arise from the arbitrary displacements of joints. These forces are applied at every

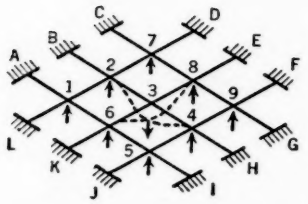


FIG. 3.—GRIDWORK WITH JOINT 3 DISPLACED

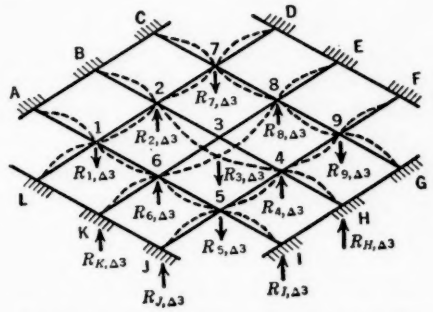


FIG. 4.—ASSUMED ELASTIC SURFACE AFTER DISTRIBUTION OF MOMENTS CAUSED BY DEFLECTION AT JOINT 3

joint, and the translations they cause or prevent must be such as to produce an equilibrium with the forces actually acting on the structure.

After the distribution factors for the unbalanced moments and torques at every joint have been determined, an arbitrary displacement (Δ) is introduced, in turn, at each singular joint as shown in Fig. 3. The singular joints of the particular grid illustrated are at points 1, 2, and 3. Deflections at all other joints will induce similar moment and torque patterns. The auxiliary force system shown consists of reactions at all joints other than joint 3 at which point the Δ_3 displacement has been produced. These reactions prevent displacement from the horizontal datum plane but do not prohibit rotation and twist in the grid members. The lengths of the grid members being equal in both directions, the deflection Δ_3 induces fixed-end moments equal to $6EI\Delta_3/l^2$ at the initially fixed joints 6, 2, 8, 4, and at joint 3 where the deflection occurs.

Moments are placed on the grid first in the east-west direction and are assigned the customary moment distribution signs as viewed from the south. Ordinary fixed-end moments need only be introduced on the beams in one direction, for the symmetry of the gridwork yields identical distribution patterns in the mutually perpendicular directions.

Next, the joints are progressively released and balanced, introducing bending moments and torques at all beam intersections. After all the joints are balanced, a carry-over process is effected, torque moments carrying over to opposite ends of the beam at their full values but with opposite sign, and bending moments carrying over as half values with the same sign. The second cycle is completed by rebalancing each joint with the usual distribution factors. Three or four cycles generally furnish a thorough dispersion of the original fixed-end moments.

When the distribution process is complete at any one joint, bending moment values at the beam ends are used to determine the reactions in terms of the Δ_3 displacement. Although a distribution is performed in the east–west direction only, a similar group of bending moments must be placed on the north–south beam in order to define the complete reaction pattern. Fig. 4 shows an assumed deflection pattern due to a Δ_3 displacement at joint 3. The notation should be self evident. For instance, $R_{9,\Delta 3}$ is the reaction at joint 9 caused by a deflection Δ_3 at joint 3. It should be remembered at this point that all reactions are expressed as coefficients of $\Delta_3 E I/l^2$. In the particular grid selection, similar displacements must be introduced at the other singular points 1 and 2, and reactions must be calculated.

Using a reciprocal relationship it is noted that the reaction at any point 4, for example, due to the deflection Δ_3 ($R_{4,\Delta 3}$) will be equal to the reaction at 3 due to the displacement Δ_4 at joint 4.

$$R_{4,\Delta 3} = R_{4,\Delta 4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Thus the complete reaction at joint 3, with all points of the grid displaced through unknown Δ values, will be:

$$R_{1,\Delta 3} + R_{2,\Delta 3} + R_{3,\Delta 3} + R_{4,\Delta 3} + R_{5,\Delta 3} + R_{6,\Delta 3} + R_{7,\Delta 3} + R_{8,\Delta 3} + R_{9,\Delta 3} \ldots (6a)$$

or

$$R_{3,\Delta 3} + R_{3,\Delta 2} + R_{3,\Delta 3} + R_{3,\Delta 4} + R_{3,\Delta 5} + R_{3,\Delta 6} + R_{3,\Delta 7} + R_{3,\Delta 8} + R_{3,\Delta 9} \ldots (6b)$$

Since only 3 singular points exist, at joints 1, 2, and 3,

$$R_{3,\Delta 1} = R_{3,\Delta 7} = R_{3,\Delta 9} = R_{3,\Delta 5} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7a)$$

$$R_{3,\Delta 2} = R_{3,\Delta 8} = R_{3,\Delta 4} = R_{3,\Delta 6} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7b)$$

Therefore, $R_{3,\Delta 3} + 4\,R_{3,\Delta 1} + 4\,R_{3,\Delta 2}$ is the total reaction influence at 3. Similar equations can be written at points on the grid and equated to a load P that might exist at the intersection, or else equated to zero if there is no load at the joint.

In the case of a uniform load across the plate, a load P exists at every intersection and the displacement of three points 1, 2, 3 will define the elastic surface. Consequently, in this simple case, only three simultaneous equations need be solved for the unknowns Δ_1, Δ_2, and Δ_3.

With the deflection values known at each point the moments and torques expressed in the distribution process in terms of Δ can be accumulated by simple multiplication and addition.

Example A—Deflections in a Simple Gridwork.—In this first application of the analysis no attempt has been made to draw an analogy between a gridwork and a slab. Fig. 5 illustrates a gridwork consisting of circular bars rigidly connected at each junction, with complete fixity against bending and twist realized at beam ends. Deflections will be determined in this structure when it is subjected to equal normal loads P at all joints and when a single load P is applied at joint 1. Since all lengths are equal and cross sections are the same, the distribution factors at each of the joints are identical. The distribution factors are computed, assuming $G = 0.36\,E$ and $I_p = 2\,I$.

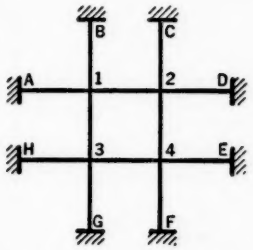


FIG. 5.—GRIDWORK OF RIGIDLY CONNECTED BEAMS

$$M_{1A} = \frac{4\,E\,I\,\theta}{l} = M_{12} \ldots\ldots\ldots (8)$$

Assuming joints A, B, 2, and 3 temporarily fixed

$$M_{1B} = \frac{G\,I_p\,\theta}{l} = \frac{0.72\,E\,I\,\theta}{l} = M_{13} \ldots\ldots\ldots\ldots\ldots (9)$$

If $\frac{E\,I\,\theta}{l}$, known to be constant, is temporarily omitted:

$$M_{1A} = \frac{4}{4 + 4 + 0.72 + 0.72} = 0.4237\,M \ldots\ldots\ldots\ldots (10)$$

in which M is an unbalanced moment released at the joint. Likewise,

$$M_{IB} = \frac{0.72}{4 + 4 + 0.72 + 0.72} = 0.0763\,M \ldots\ldots\ldots\ldots (11)$$

Similar equations can be set up for the remaining points on the gridwork. The percentage of the unbalanced moment resisted by bending moments developed in beams is 42.37%, and the percentage resisted by torque moments in the north-south beams is 7.63%.

When joint 1 is displaced a distance Δ_1, fixed-end moments of $6\,E\,I\Delta_1/l^2$ are induced at points A, 1, and 2 along the beam A D and also at 3, 1, and B along the beam C B. Fig. 6 shows the fixed-end moments multiplied by 1,000 but exclusive of the constant quantity $E\,I\,\Delta_1/l^2$ along the east-west beams, together with the ensuing moment and torque distribution. The signs of the moments are in accordance with the moment distribution convention when viewed from the south. Moments are not distributed in the north-south direction since they are identical to those developed in the east-west beams.

Equilibrating reactions are derived from the end moments in the east-west direction obtained by the distribution process. The equations of static equilibrium are used to determine these reactions after the beams are subdivided into their component lengths, as shown in Fig. 7. Because of the constant beam length l, the reaction values shown are multiples of the term $E\,I\,\Delta_1/l^3$. When an equal series of reactions derived from the north-south end moments are superimposed upon the grid and algebraically added to the first group of reactive forces, the equilibrated system shown in Fig. 8(a) results.

By the theorem of reciprocity, the reaction $13.188\ E\ I\ \Delta_1/l^3$ at junction 2, caused by the deflection Δ_1 at joint 1, is equal to the reaction at joint 1 caused

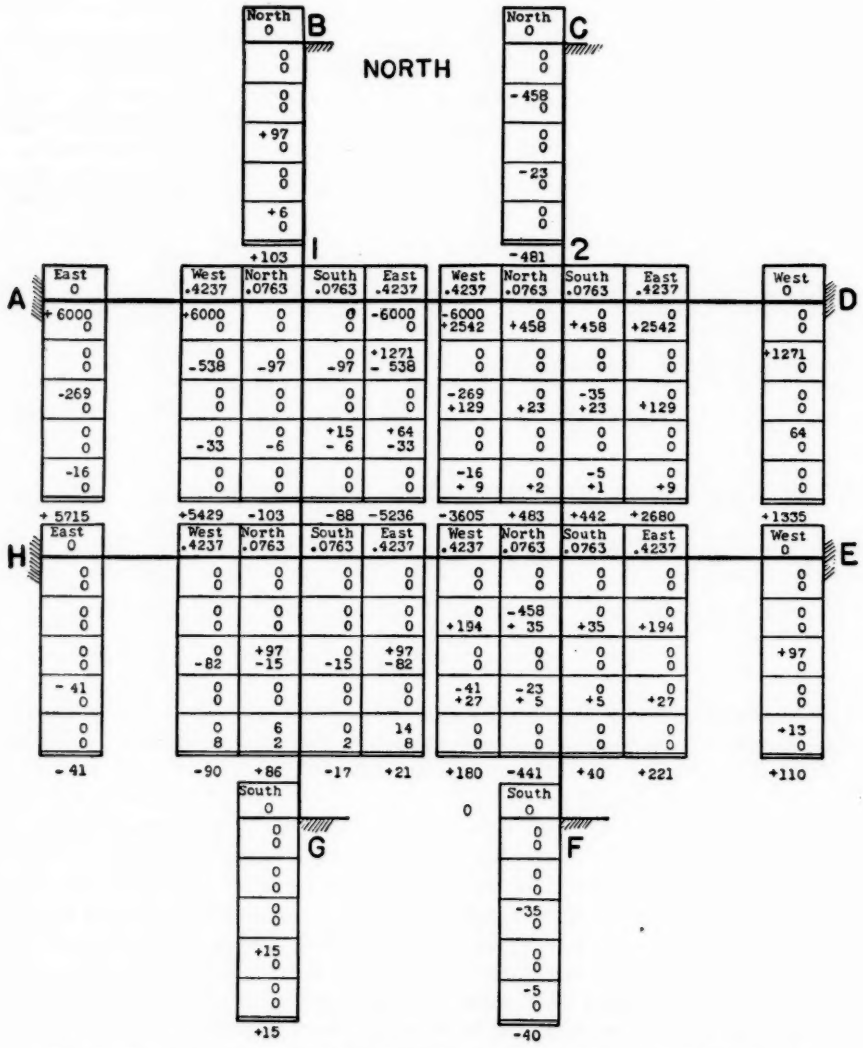


FIG. 6.—DISTRIBUTION OF EAST-WEST, FIXED-END MOMENTS. DISPLACEMENT AT JOINT 1

by an equal deflection Δ_2 at joint 2. If a similar reasoning is employed at all grid points, the total reaction at joint 1 can be written as follows:

$$R_1 = 39.970\ E\ I\ \Delta_1/l^3 - 13.188\ E\ I\ \Delta_2/l^3 - 0.260\ E\ I\ \Delta_4/l^3 - 13.188\ E\ I\ \Delta_3/l^3 \quad (12)$$

If every grid joint supports a load of P, the following expression can be obtained by equating vertical forces to zero:

$$P = 39.970\,E\,I\,\Delta_1/l^3 - 13.188\,E\,I\,\Delta_2/l^3 - 0.260\,E\,I\,\Delta_4/l^3 - 13.188\,E\,I\,\Delta_3/l^3 \quad \ldots\ldots\ldots\ldots (13)$$

But with equal forces P located at all joints of the symmetrical grid:

$$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

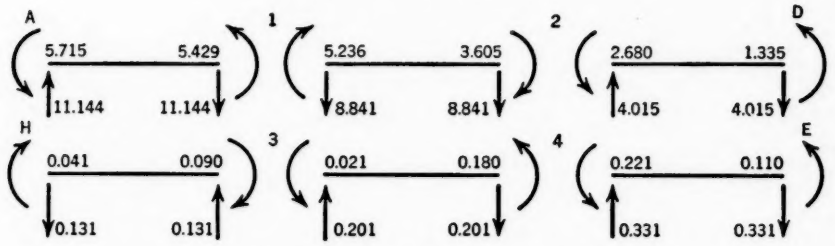


FIG. 7.—MOMENTS AND REACTIONS ON GRID BEAMS

Therefore:

$$13.332\,E\,I\,\Delta_1/l^3 = P \quad \ldots\ldots\ldots\ldots\ldots\ldots (15a)$$

or transposing

$$\Delta_1 = \frac{P\,l^3}{13.332\,E\,I} = \frac{0.075\,P}{l^3\,E\,I} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15b)$$

Fig. 8(b) illustrates the deflected grid subjected to equal loads P at all joints. The deflections are expressed as coefficients of $\frac{P\,l^3}{E\,I}$.

When a single concentrated load P is applied at joint 1, the deflections developed at joints 1, 2, 3, and 4 are not equal. By equating vertical forces at all grid joints to zero the following linear equations are developed:

$$39.970\,E\,I\,\Delta_1/l^3 - 13.188\,E\,I\,\Delta_2/l^3 - 0.260\,E\,I\,\Delta_4/l^3 - 13.188\,E\,I\,\Delta_3/l^3 = P \quad \ldots\ldots\ldots (16a)$$

$$-13.188\,E\,I\,\Delta_1/l^3 + 39.970\,E\,I\,\Delta_2/l^3 - 13.188\,E\,I\,\Delta_4/l^3 - 0.260\,E\,I\,\Delta_3/l^3 = 0 \quad \ldots\ldots\ldots (16b)$$

$$-0.260\,E\,I\,\Delta_1/l^3 - 13.188\,E\,I\,\Delta_2/l^3 + 39.970\,E\,I\,\Delta_4/l^3 - 13.188\,E\,I\,\Delta_3/l^3 = 0 \quad \ldots\ldots\ldots (16c)$$

$$-13.188\,E\,I\,\Delta_1/l^3 - 0.260\,E\,I\,\Delta_2/l^3 - 13.188\,E\,I\,\Delta_4/l^3 + 39.970\,E\,I\,\Delta_3/l^3 = 0 \quad \ldots\ldots\ldots (16d)$$

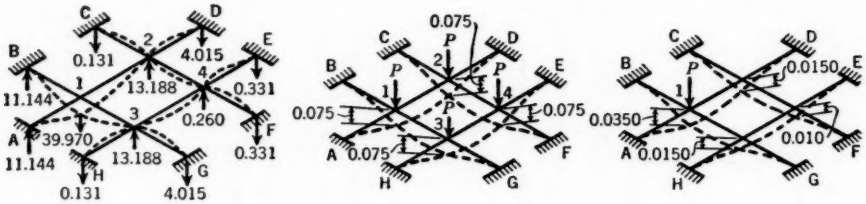
Solved simultaneously, the equations yield the following values for deflection:

$$\Delta_1 = 0.0350\,P\,l^3/E\,I$$
$$\Delta_2 = 0.0150\,P\,l^3/E\,I$$
$$\Delta_3 = 0.0101\,P\,l^3/E\,I$$
$$\Delta_4 = 0.0150\,P\,l^3/E\,I$$

Fig. 8(c) shows the deflections as coefficients of $P\,l^3/E\,I$ at each of the four joints.

Since the bending and the torsional moments have been determined in terms of the various displacements, it is a simple matter to calculate the moments after the deflections have been determined.

From Fig. 6 the moment at A due to displacement at joint 1 is $+ 5{,}715\ E\ I\ \Delta_1/l^2$ and the torque along beam A 1 is $103\ E\ I\ \Delta_1/l^2$ units, as shown at B. The torque of $103\ E\ I\ \Delta_1/l^2$ could be calculated directly by distributing fixed-end moments in the north-south direction, but this procedure is unnecessary due to symmetry. Furthermore, with a displacement at joint 2,



(a) ASSUMED ELASTIC SURFACE AFTER DISTRIBUTION OF MOMENTS DUE TO Δ_1

(b) DEFLECTION SURFACE LOAD P AT JOINTS 1,2,3,4

(c) DEFLECTION SURFACE LOAD P AT JOINT 1

FIG. 8.—DEFLECTIONS IN A SIMPLE GRIDWORK

the moment $+ 1{,}335\ E\ I\ \Delta_1/l^2$ shown at point D in the figure would develop as $- 1{,}335\ E\ I\ \Delta_2/l^2$ at point A. The change in sign arises because original fixed-end moments should be introduced as $- 6{,}000\ E\ I\ \Delta_2/l^2$. With displacements at all joints the total moment at A is:

$$M_{A\,1} = + 5{,}715\ E\ I\ \Delta_1/l^2 - 1{,}335\ E\ I\ \Delta_2/l^2 - 41\ E\ I\ \Delta_3/l^2 - 110\ E\ I\ \Delta_4/l^2 \quad (17a)$$

Substituting values for Δ as previously determined,

$$M_{A\,1} = + 200.03\ P\ l - 17.03\ P\ l - 0.62\ P\ l - 1.11\ P\ l \ldots\ldots (17b)$$

$$M_{A\,1} = + 181.27\ P\ l \ldots\ldots\ldots\ldots\ldots\ldots (17c)$$

in which the positive sign is based on moment distribution convention. All other moments and torques can be determined in like fashion.

Example B—Analysis of a Slab, With and Without a Center Support.—As a second illustration of the applicability of the method, the 1/16 in.-thick slab shown in Fig. 9 has been selected for analysis. A combination of fixed and free boundaries has been chosen for the slab in order to prove that complexities arising from variable edge conditions are easily handled. The effects of continuity will be studied by determining slab deflections both with and without a center vertical support at joint 12.

If the slab shown is divided into a series of 3-in. wide strips, each strip can be considered a beam of an equivalent grid system. The grid beams and their centerlines, along which the deflections will be determined, are shown in Fig. 9. Because of the fixed- and free-end conditions, three sets of distribution factors are required for the distribution of moments and torques over the gridwork.

Interior joints require but one set of distribution constants, for when a displacement is introduced two equal length and equal cross-section beams

twist through an angle as two identical but perpendicular beams rotate through the same angle. The same stiffness factors are developed, and consequently the same distributive effects are required for the distribution of fixed-end moments in the perpendicular direction. Edge joints require different constants in the two directions, for when moments are distributed int he north-south direction, one east-west beam is subjected to torque; and when moments are distributed in the east-west direction two north-south beams are subjected to twist. The values of the distribution factors for interior and exterior joints are shown in Fig. 10. These constants can be computed by assuming all beams to be of constant length l, and also assuming that $G = 0.377\ E$, and that the ratio of beam depth to beam width = 1/16 in. to 3 in. = 1/48. The expression for moment required to produce a unit angle of twist in a rectangular beam has been taken from Mr. Timoshenko.[9]

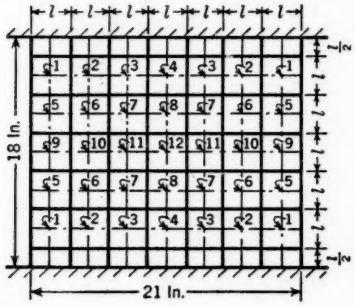


FIG. 9.—DESIGNATION OF GRID JOINTS ON A PLATE

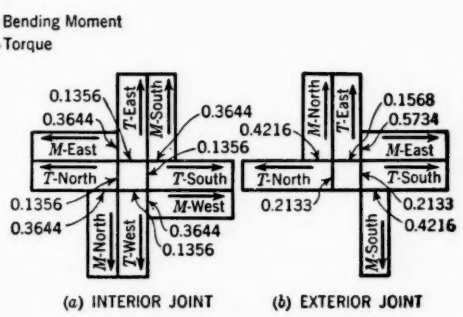


FIG. 10.—DISTRIBUTION FACTORS AT TYPICAL JOINTS

When the slab is subjected to a uniform vertical load of w lb per sq in., the equivalent gridwork is loaded at each joint with a concentration $P = w\ l^2$. Twelve deflections, Δ_1 through Δ_{12} at the numbered joints shown in Fig. 9, will completely define the elastic surface. Consequently, equal but arbitrary displacements are introduced at each of the similarly numbered singular grid joints in turn, while the remaining joints are temporarily fixed on a horizontal datum plane by vertical reactions. When all similarly numbered joints are displaced simultaneously, a symmetrical distribution results and the computation can be confined to one quarter of the slab.

Inasmuch as the numerical process of distribution is similar at all joints, the procedure is illustrated for displacements at joint 9 only, as shown in the chart of Fig. 11. This chart, showing only the north-west quarter of the slab, facilitates the simultaneous distribution of the north-south and east-west, fixed-end moments of $6\ E\ I\ \Delta_9/l^2$ developed at joints 9, 5, and 10. The simultaneous distribution is necessitated by the different end conditions in the two directions. For convenience, moments are represented by the constant coefficient 6 multiplied by 1,000. As in example A, the carry-over factors for bending moment and torsional moment are 1/2 and -1, respectively. The singular

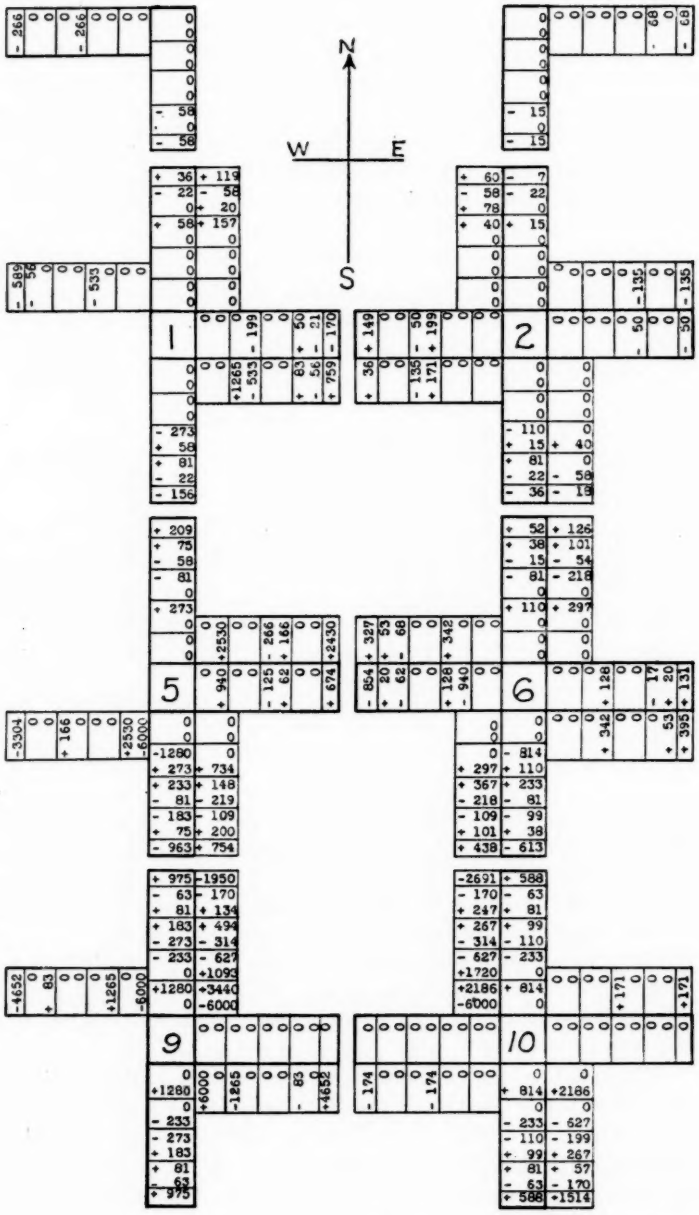
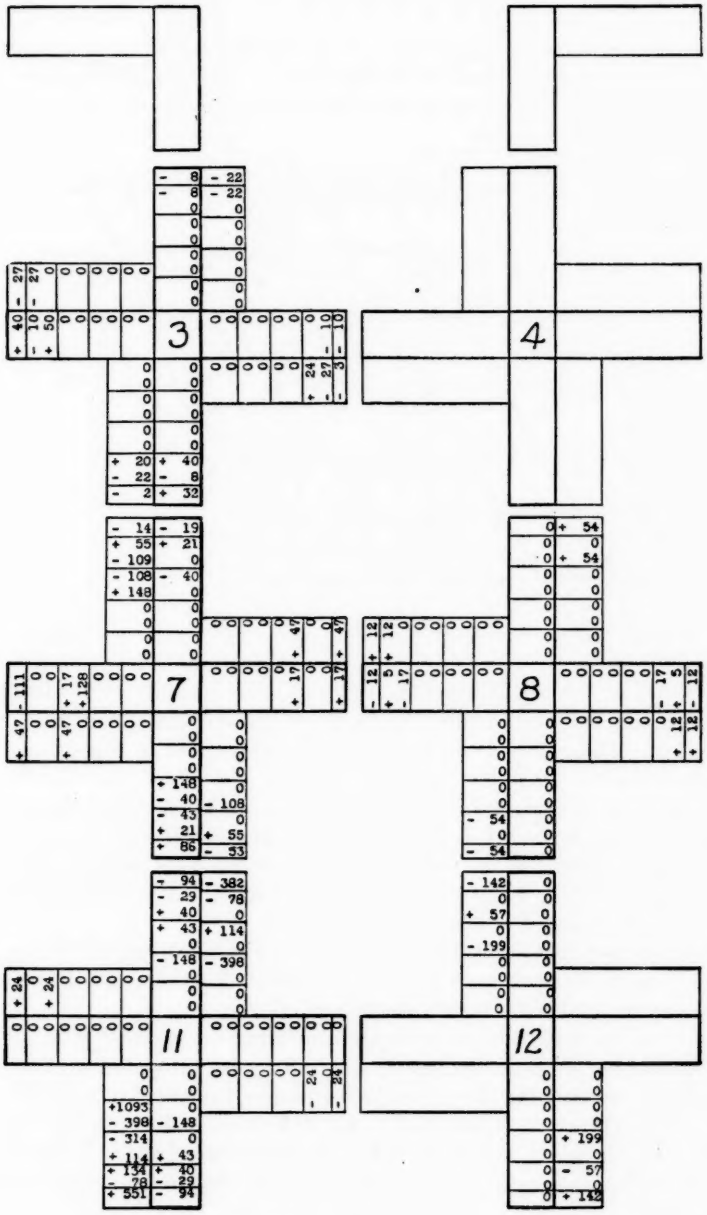


FIG. 11.—DISTRIBUTION OF FIXED-END



MOMENTS. DISPLACEMENT Δ_9 AT JOINT 9

joint reactions, exclusive of $E\,I\,\Delta_9/l^3$ caused by the Δ_9 displacements are illustrated in Fig. 12.

Similar distributions are performed at each of the 12 singular joints, each operation reducing by two the number of unknowns. The complete process reduces from 36 to 12 the number of linear equations to be solved simultaneously.

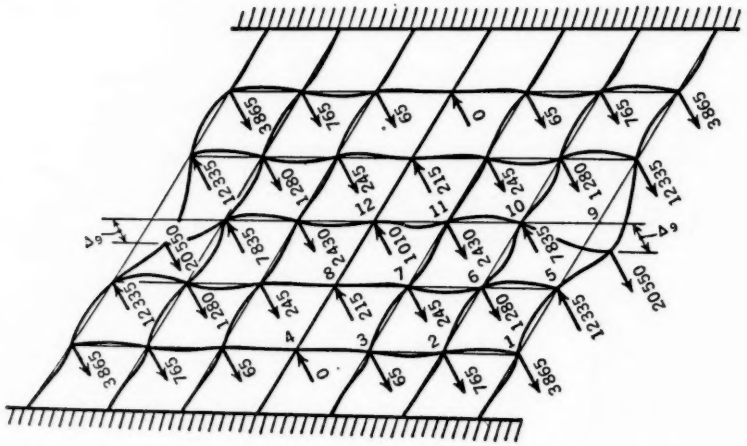


FIG. 12.—ASSUMED ELASTIC SURFACE AFTER DISTRIBUTION OF MOMENTS CAUSED BY DEFLECTION Δ_9

After the effects of the displacements at each of the singular joints have been determined, the reactions shown in Table I are calculated, using the methods of statics. The reactions caused by Δ_9 are illustrated in Fig. 12 and entered in Table 1(*a*).

When the reactions—expressed in terms of the unknown Δ displacements but still exclusive of the term $E\,I/l^3$—are accumulated at each joint and each reaction equated to a vertical load 1,000 P, the twelve equations used to determine the twelve Δ's result. Since the equations are merely a restatement of Table I, they are not repeated at this point.

With a center support at joint 12, only the first eleven equations are required, inasmuch as Δ_{12} is known to be equal to zero. Furthermore, the Δ_{12} term is not present in the eleven equations solved.

The solution of these equations is facilitated by using the method of M. H. Doolittle.[10] The values for deflection at the grid joints are tabulated in Table 1(*b*). Since 1,000 P was used in the equation formation, the effect of multiplying $6\,E\,I\,\Delta/l^3$ by 1,000 was obliterated. All deflection values shown are multiples of $P\,l^3/E\,I$, in which the E, I, and l are properties of the grid beam.

As in example A, moments and torques may be determined after the deflections are known.

Example C—Analysis of a Uniformly Loaded Square with Clamped Edges.—The square slab shown in Fig. 13 has been selected for analysis to show the

[10] "Application of the Theory of Least Squares to the Adjustment of Triangulation," by Oscar S. Adams, *Special Publication No. 28*, U. S. Coast and Geodetic Survey, Washington, D. C., 1915.

TABLE 1.—REACTIONS AND DEFLECTIONS AT SINGULAR JOINTS CAUSED BY Δ DISPLACEMENTS

Joint number	Displacement												
	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}	P
(*a*) SUMMATION OF REACTIONS IN TERMS OF $EI\Delta/l^3$													
1	+24,790	− 7,470	+ 2,490	− 480	−14,030	+ 860	+ 180	− 105	+ 3,865	+ 765	+ 140	− 30	= +1,000
2	− 7,470	+35,610	−12,100	+ 3,205	+ 860	−14,350	0	+ 410	+ 765	+ 3,225	+ 585	+ 120	= +1,000
3	+ 2,490	−12,100	+40,800	−11,850	+ 190	0	−13,215	+ 15	+ 65	+ 530	+ 3,505	+ 515	= +1,000
4	− 960	+ 6,410	−23,700	+37,600	− 230	− 830	0	−13,625	0	+ 250	+ 1,030	+ 3,390	= +1,000
5	−14,030	+ 860	+ 190	− 115	+24,450	− 7,230	+ 2,530	− 525	−12,335	+ 1,280	+ 300	− 145	= +1,000
6	+ 860	−14,350	0	+ 415	− 7,230	+35,420	−12,050	+ 3,330	+ 1,280	−13,040	+ 340	+ 460	= +1,000
7	+ 180	0	−13,215	0	+ 2,530	−12,050	+40,690	−11,790	+ 245	+ 450	−11,850	+ 325	= +1,000
8	− 210	+ 820	+ 15	−13,625	− 1,050	+ 6,660	−23,580	+37,355	− 215	+ 830	+ 645	−12,330	= +1,000
9	+ 7,730	+ 1,530	+ 130	0	−24,670	+ 2,560	+ 490	− 215	+20,550	− 7,835	+ 2,430	− 505	= +1,000
10	+ 1,530	+ 6,450	+ 1,060	+ 250	+ 2,560	−26,080	+ 450	+ 830	− 7,835	+31,900	−12,575	+ 3,215	= +1,000
11	+ 140	+ 1,170	+ 1,030	+ 1,030	+ 600	+ 680	−23,700	+ 645	+ 2,430	−12,575	+37,240	−12,330	= +1,000
12	− 120	+ 480	+ 2,060	+ 6,780	− 580	+ 1,840	+ 1,300	−24,660	− 1,010	+ 6,430	−24,660	+33,970	= +1,000
(*b*) VALUES OF DEFLECTIONS IN TERMS OF Pl^3/EI													
Without support	1.04	1.01	0.98	0.97	2.69	2.60	2.52	2.51	3.41	3.33	3.21	3.20	....
With support	0.78	0.58	0.33	0.21	1.86	1.36	0.64	0.29	2.27	1.59	0.68	0.00	....

accuracy of the grid method as used to define the entire elastic surface of a loaded slab. The characteristics of this slab are as follows: $G = 4{,}000{,}000$; $E = 10{,}600{,}000$; $\mu = 0.33$; $a = 2$ in.; $b = 1/16$ in. The slab has a uniform load of w lb per unit area. The dashed lines in Fig. 13 represent the grid beam centerline. Deflection values obtained at ten different points on the

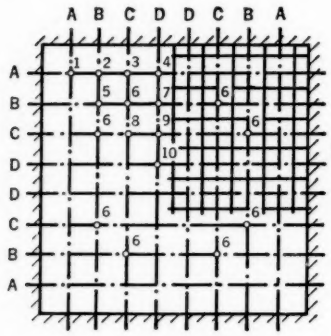


FIG. 13.—SELECTION OF GRIDWORK

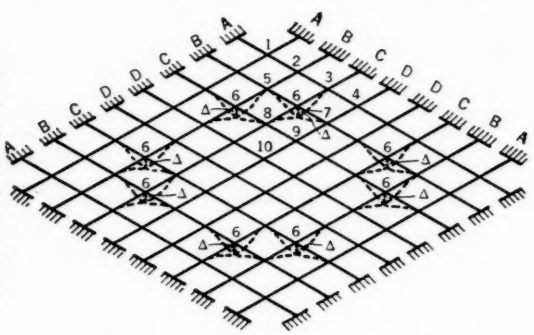


FIG. 14.—SIMULTANEOUS DISPLACEMENT OF JOINT 6

slab will be compared with the exact values obtained by the mathematical theory of elasticity.[11]

Nine grid beam centerlines, each two inches apart, are selected in the two orthogonal directions, in order to convert the slab into a pseudogrid system. This slab division gives rise to the ten singular joints shown. Under the action of a uniform load, the square slab will deflect symmetrically around its north–south, east-west and diagonal axes. Because of this symmetry, the 10 singular joints contained within any one octant of the grid will completely define the elastic surface.

Since the grid beams in both directions have identical lengths and cross sections, the resistances to bending and to twist will be constant at every joint. When the physical characteristics of the slab and the constant length and cross-sectional properties of the grid beams are involved in the proper expression,[10] the following distribution factors will result:

$$\text{Percentage moment} = 13.49 \text{ torque} \quad \ldots\ldots\ldots\ldots\ldots (18a)$$

$$\text{Percentage moment} = 36.51 \text{ bending} \quad \ldots\ldots\ldots\ldots\ldots (18b)$$

If an unbalanced moment M is applied on any joint of the gridwork in either direction, 36.51% of the applied moment is resisted by two beams in bending, and two beams in torsion resist 13.49% of the applied moment.

The multiple symmetry existent in this example permits the simultaneous displacement of all similarly numbered joints and the calculation of the auxiliary force systems developed by these displacements. As an example, the eight joints designated "6" in Fig. 13 will be deflected through equal distances when the gridwork is uniformly loaded. Furthermore, these eight

[11] "The Calculation of Maximum Deflection, Moment, and Shear for Uniformly Loaded Rectangular Plate with Clamped Edges," by I. A. Wojtaszak, *Transactions*, American Society of Mechanical Engineers, Vol. 59, 1937, p. A-173.

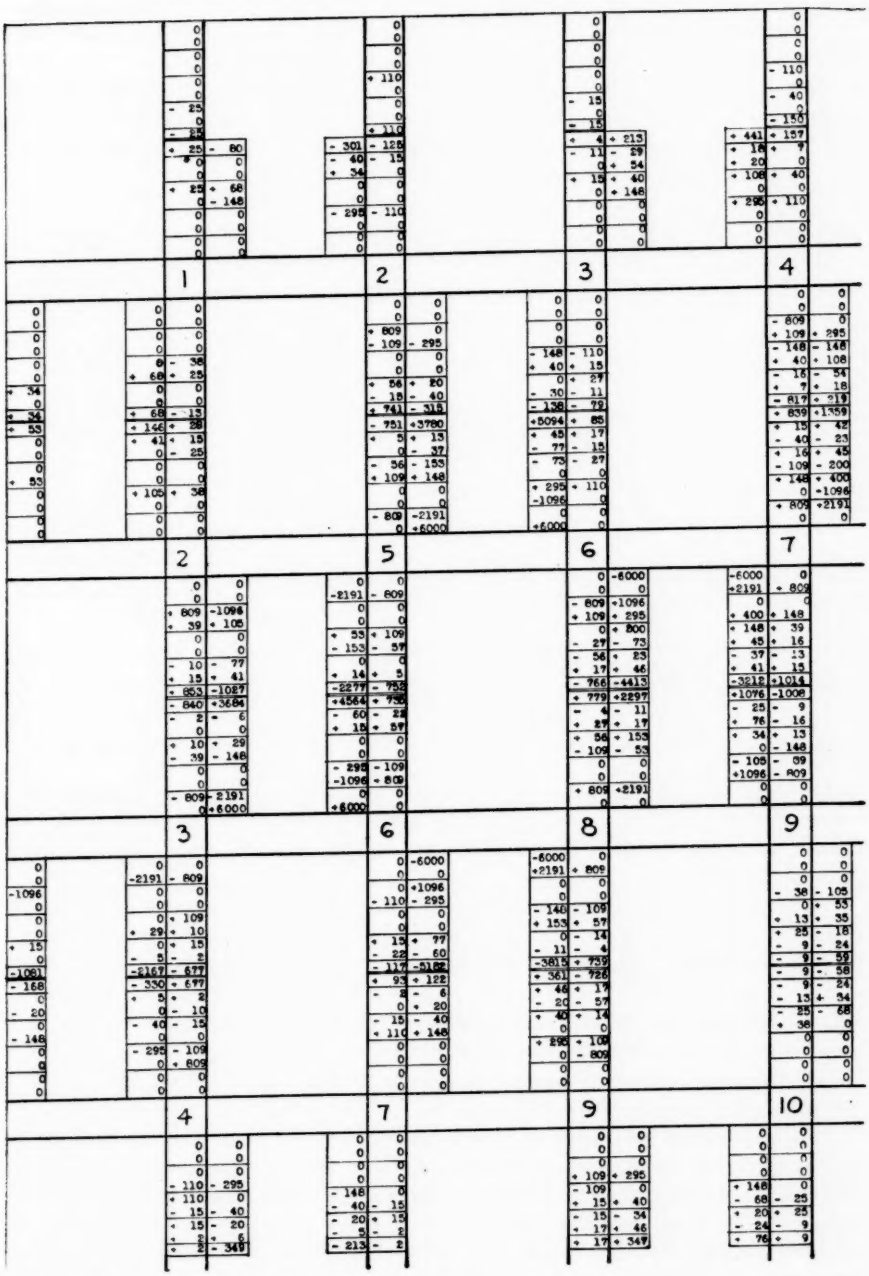


FIG. 15.—DISTRIBUTION OF EAST-WEST, FIXED-END MOMENTS CAUSED BY DISPLACEMENT Δ_6 AT JOINT 6

joints form one of a series of singular joints. They are singular because their displacement, when coupled with a prohibition of displacement at all other joints, will develop a distinctive moment and torque pattern.

When all of the joints designated as "6" are simultaneously displaced, as shown in Fig. 14, a symmetrical distribution of bending and torsional moments will result from the symmetrically located fixed-end moments. Consequently the computations can be confined to the upper left quadrant.

The numerical process of distribution is illustrated in Fig. 15 for the displacements of joints 6 only. Again, the fixed-end moments are expressed as multiples of $E\,I/1{,}000\,l^2$, and the signs of these moments are in keeping with the usual moment distribution conventions when viewed from the south.

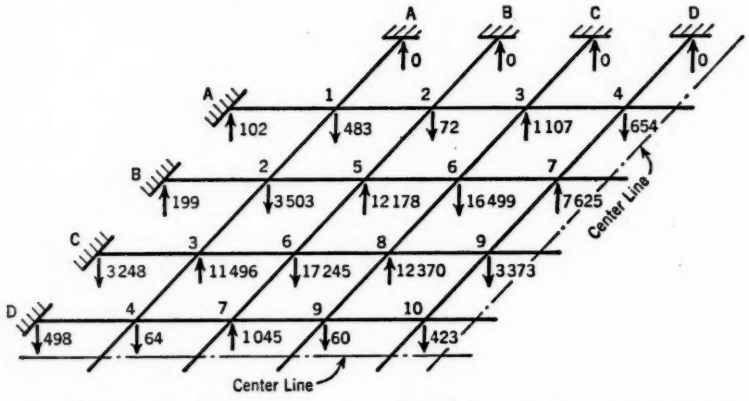


FIG. 16.—AUXILIARY FORCE PATTERN IN NORTH-WEST QUADRANT CAUSED BY DEFLECTION Δ_6

Only the east-west, fixed-end moments are considered on the grid system because the symmetry of the structure allows identical force patterns to be developed in the north-south direction, when north-south, fixed-end moments are distributed.

After the distribution process is complete, final bending moment values at the grid beam ends are used to compute end shear values in terms of the original Δ_6 displacements. The force pattern developed in the north-west quadrant is shown in Fig. 16. This force system is not complete, however, because it shows only the effects of the east-west, fixed-end moments induced by the displacements Δ_6. Fortunately, the force pattern developed by the north-south, fixed-end moments can be expressed without further calculation. For example, the reactions + 102, − 483, − 72, + 1,107, − 654, read from east to west along grid beam A shown in Fig. 16, will also occur in the north-south direction along the orthogonal grid beam A. The sixth column of Table 2(*a*) shows the total reaction values due to the simultaneous displacements. The total reaction values result from an algebraic summation of reaction magnitudes developed by both north-south and east-west, fixed-end moments.

Similarly, the remaining 9 groups of singular joints are displaced, the fixed-end moments distributed, and the auxiliary forces calculated. These

TABLE 2.—SIMULTANEOUS EQUATIONS AND SOLUTIONS FOR DEFLECTION, UNIFORMLY LOADED SQUARE PLATE

At joint	Δ1	Δ2	Δ3	Δ4	Δ5	Δ6	Δ7	Δ8	Δ9	Δ10	External load
(*a*) TOTAL REACTION VALUES EXPRESSED IN TERMS OF $K\,I\,\Delta/l^3$ =											
1	−40,870	+25,000	− 6,764	+ 1,090	+ 2	− 966	+ 266	− 120	+ 32	0	−1,000*P*
2	+12,500	−37,559	+11,289	− 2,512	+12,924	− 3,575	+ 280	− 511	+ 29	+ 17	−1,000*P*
3	− 3,382	+11,290	−36,942	+ 8,597	− 191	+12,603	− 483	− 3,368	− 58	+ 148	−1,000*P*
4	+ 543	− 2,513	+ 8,599	−25,590	− 291	− 718	+12,808	− 631	− 3,745	+ 703	−1,000*P*
5	0	+25,850	− 380	− 586	−34,130	+24,356	− 5,238	− 384	− 638	− 88	−1,000*P*
6	− 484	− 3,575	+12,605	− 718	+12,180	−33,744	+ 8,672	+12,375	− 3,438	− 419	−1,000*P*
7	+ 134	+ 281	− 478	+12,806	− 2,619	+ 8,670	−21,834	− 835	+11,437	−2,937	−1,000*P*
8	− 120	− 1,022	− 6,734	− 1,260	− 382	+24,740	− 1,668	−32,598	+18,396	−1,410	−1,000*P*
9	+ 16	+ 28	− 59	− 3,743	− 319	− 3,433	+11,436	+ 9,197	−22,204	+8,215	−1,000*P*
10	0	+ 34	+ 300	+ 1,406	− 86	− 846	− 5,970	− 1,404	+16,428	−9,646	−1,000*P*
(*b*) SOLUTIONS											
...	0.272	0.766	1.177	1.395	2.135	3.304	3.935	5.135	6.134	7.341	....

operations are not included because of their repetitious character. All auxiliary force values, expressed as multiples of $E\,I\,\Delta/1{,}000\,l^3$, are shown on the left side of Table 2.

The ten linear equations defining the elastic surface of the slab under load are found by equating the 10 separate reaction magnitudes at each joint to the externally applied load P at the same joint. Consequently, Table 2 is not only an expression of auxiliary force systems but also a statement of the ten linear equations of equilibrium.

The force P is everywhere equal to $w\,l^2$. For convenience, the load P is multiplied by 1,000, thus nullifying the effect of the thousandfold increase in moment value. The solutions of these equations are shown in Table 2(*b*) in which the values of displacements are multiples of $P\,l^3/E\,I$.

TABLE 3.—COMPARISON OF DEFLECTIONS IN TERMS OF $w\,a^4/E\,h^3$

Method of calculation	1	2	3	4	5	6	7	8	9	10
Method of I. A. Wojtaszak	.00046	.00133	.00204	.00240	.00371	.00592	.00691	.00905	.01087	.01308
Grid analogy	.00049	.00140	.00215	.00255	.00390	.00604	.00720	.00939	.01122	.01343

In this classical slab-problem, the magnitudes of deflections over the loaded surface, as determined by the theory of elasticity and by the grid analysis procedure, are found to be in excellent agreement. A comparison of deflections at the ten selected positions is shown in Table 3. The results

obtained by the theory of elasticity, as expressed by I. A. Wojtaszak,[11] are in terms of $w\,a^4/E\,h^3$, in which a = side length of square slab; h = thickness of slab; and the other terms are as previously defined.

Since the length of the grid beam (l) in this example is equal to $a/9$, the grid analysis values shown in Table 2 can be converted as follows:

$$\Delta_6 = 3.304\,\frac{P\,l^3}{E\,I} = \frac{3.304\,(w\,l^2)\,l^3}{E\,l\,h^3/12} = \frac{3.304\,(12)\,w\,l^4}{E\,h^3}$$

$$= \frac{3.304\,(12)\,w\,a^4}{9^4\,E\,h^3} = \frac{0.0604\,w\,a^4}{E\,h^3}.$$

Along the slab boundary there is less grid beam definition since the nearest grid beam is one half a grid beam length removed from the edges. As a result the greatest error in the grid analysis occurs near the slab boundary.

Conclusions

The analysis of gridworks and slabs subjected to normal load is a tedious process, for the calculation of deflections and the determination of their significance should be performed at many points in order to define clearly the resulting elastic action. Neither the exact mathematical methods nor the approximation schemes can circumvent this difficulty.

The procedure developed in this paper should compare favorably with any known method for determining slab and gridwork deflections. The method is fundamentally a numerical process that results in the solution of only one series of simultaneous equations. When any series of linear equations contain frequent coefficients of zero values for the unknowns, the solution of the equations is relatively simple. This condition may be realized in the slab deflection equations of this method by performing fewer cycles of distribution, thus confining the dispersed moments to fewer joints. Obviously, more errors are incurred when this expediency is employed.

When applied to a true gridwork, the outlined method results in an "exact" solution, for no degree of indeterminateness is avoided. The moment and torque distributions express the unknown moments and twists at each joint in terms of the unknown displacements that comprise the terms of the linear equations. It is believed that the initial threefold indeterminacy assumed at each joint represents a structural condition more realistic than ordinarily considered by other methods of grid analysis.

While the authors can offer no mathematical proof of the analogy between slab and gridwork, numerous results computed by the grid method and checked by the theory of elasticity seem to validate the grid analysis procedure. Obviously, the accuracy of any analysis is dependent on the number of grid beams used to approximate the slab. Using an analogous grid consisting of six north-south beams and six east-west beams and solving directly linear equations containing all three unknowns at each joint, Béla Enyedi[6] has shown that the computed center moment on a uniformly loaded square plate agrees remarkably well with Henri Marcus' solution by the theory of elasticity. It is interesting to note that the deflections in example C, computed by the grid

method, agree more closely with the exact values than those offered by Mr. Marcus' finite difference analysis[12] using the same slab division.

The method of moment and torque distribution for determining slab and gridwork deflections is relatively simple to apply to a wide variety of problems, heretofore difficult to solve.

The degree and the variation of fixity or freedom existing at the slab or gridwork boundary does not complicate the solutions for deflections and moments, since the distribution factors at grid beam terminals are so assigned as to reproduce the given boundary condition. When the fixed boundary conditions of the slab in example C were imposed on the grid beam ends, the elastic surface developed by the grid system was found to be strikingly similar to that computed by Mr. Wojtaszak. Fig. 17 illustrates a slab irregular in shape and variable in boundary conditions and the possible arrangement of grid beams defining the various conditions.

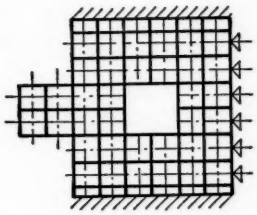


FIG. 17.—GRID BEAMS FOR VARIABLE BOUNDARY CONDITIONS

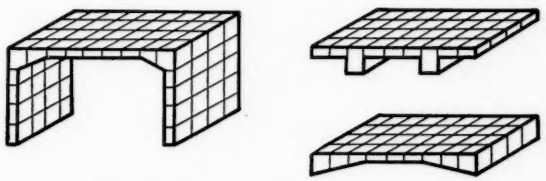


FIG. 18.—SUGGESTED GRID BEAM LAYOUTS

The analysis of flat slabs that are continuous over column or wall supports is possible by this method. As shown in example B, the center column support actually simplifies the analysis of the flat slab by prohibiting deflection at the support joint and consequently reducing the number of unknowns.

Slabs of variable cross sections, when divided into analogous grid beams, submit to solutions if the variation in distribution factors is considered. Tentative grid beam layouts for various cases are shown in Fig. 18.

Other extensions of the moment and torque distributions are feasible. Investigations are being made of slabs on elastic foundations in order to predict deflections and stresses in the slabs.

In general, the moment distribution method for determining deflections and moments in gridworks and slabs exhibits a versatility that is not characteristic of other analytical methods. There are some who will reject the simplicity of the numerical approximation for the elegance of the exact mathematical analysis. Where solutions by the theory of elasticity are known, this attitude seems justified, although a thorough evaluation of slab action by any scheme is a time-consuming process. The real merit of the grid analogy approximation used in this paper has yet to be explored on slabs heretofore defying solution.

[12] "Die Theorie elasticher Gewebe und ihre Anwendung auf die Berechnung biegsamer Platten," by Henri Marcus, Julius Springer, Berlin, 1924, p. 152.

APPENDIX.—COMPUTATION OF DEFLECTIONS

I. A. Wojtaszak gives the following expressions for the calculation of deflection of a uniformly loaded rectangular plate with clamped edges:

$$w = \frac{P}{8\,D}\left(\frac{a^2}{4} - x^2\right)\left(\frac{b^2}{4} - y^2\right) - \frac{P\,a^3}{16\,\pi^2\,D}\left\{\sum_{n=1,3}^{\infty}\frac{A_n}{n^2}\,Y_n\,\frac{\cos\dfrac{n\,\pi\,x}{a}}{\cosh^2\dfrac{n\,\pi\,b}{2\,a}}\right.$$

$$\left. + \sum_{n=1,3}^{\infty}\frac{B_n}{n^2}\,\frac{b^3}{a^3}\,X_n\,\frac{\cos\dfrac{n\,\pi\,y}{b}}{\cosh^2\dfrac{n\,\pi\,a}{2\,b}}\right\} \qquad (19)$$

in which w is the deflection; P is the load per unit area; a $(= b)$ is the length of side; x and y are coordinates of plate, origin at center; D is the plate modulus $\left[= \frac{E\,H^3}{12\,(1-\mu^2)}\right]$; μ is Poisson's ratio $(= 0.33)$; $X_n = a \sinh\frac{n\,\pi\,a}{2\,b}\cosh\frac{n\,\pi\,x}{b} - 2\,x\cosh\frac{n\,\pi\,a}{2\,b}\sinh\frac{n\,\pi\,x}{b}$; and $Y = b\sinh\frac{n\,\pi\,b}{2\,a}\cosh\frac{n\,\pi\,y}{a} - 2\,y\cosh\frac{n\,\pi\,b}{2\,a}\sinh\frac{n\,\pi\,y}{a}$ for $b/a = 1.00$.

n	A_n	B_n
1	+ 1.4138	+ 1.4138
3	+ 0.0943	+ 0.0943
5	− 0.1096	− 0.1096
7	+ 0.0765	+ 0.0765
9	− 0.0523	− 0.0523
11	+ 0.0365	+ 0.0365

The deflection values shown in Table 3 were calculated using these expressions. Deflection influences can be determined for any point defined by the coordinates x and y by expanding the above series. Since the series converges very rapidly, only three or four terms were deemed necessary. Throughout the calculation an accuracy of five decimal places was used.

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